

Newtonian wormholes with spherical symmetry and tidal forces on test particles

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A spherically symmetric wormhole in Newtonian gravitation in curved space, enhanced with a connection between the mass density and the Ricci scalar, is presented. The wormhole, consisting of two connected asymptotically flat regions, inhabits a spherically symmetric curved space. The gravitational potential, gravitational field and the pressure that supports the fluid that permeates the Newtonian wormhole are computed. Particle dynamics and tidal effects in this geometry are studied. The possibility of having Newtonian black holes in this theory is sketched.

1. Introduction

The idea that Newton's theory of gravitation, can be formulated in curved space has been recently analyzed by Abramowicz and collaborators^{1,2}. In addition to considering Newtonian gravitation in curved space, an equation linking the geometry of the 3-space with the matter sector was put forward^{3,4}. This is called an enhanced Newtonian gravitation. In particular, this modification to Newton's theory allows the construction of a wormhole space⁵. Wormholes in general relativity have been studied in several works see⁶⁻¹⁰ for example. In this paper, we report on the construction of a spherically symmetric wormhole⁵ in this enhanced Newtonian gravitation. We further analyze the tidal effects that emerge from the gravitational forces and the curvature of space in this spherically symmetric Newtonian wormhole space.

The outline of the paper is as follows: in Sec. 2 we start by writing the equations that define the enhanced Newtonian theory of gravitation. In Sec. 3 we construct a static spherically symmetric Newtonian wormhole, we find the gravitational field, the gravitational potential and the pressure of the fluid that supports the wormhole. Then, we study test particle's motion in the wormhole geometry and gravitational field. Finally we analyze the tidal forces exerted by the gravitational field and the curvature of space in two nearby particles. In Sec. 4, we conclude and speculate on

the possible existence of truly Newtonian black holes in this enhanced Newtonian theory of gravitation.

2. The fundamental equations

The classical formulation of Newton's theory of gravitation was constructed for an absolute 3-dimensional Euclidean space. We see, however, that the set of equations that comprise Newton's theory of gravitation are also well defined for curved space. Indeed, Abramowicz et al.^{1,2} recently proposed a formulation of Newton's theory of gravitation in curved space.

Poisson's equation in curved space is given by

$$g^{ij} \nabla_i \nabla_j \phi = 4\pi G \rho, \quad (1)$$

where g_{ij} is the curved space metric, ∇_i is the covariant derivative induced by the metric, the indices i, j run as $i, j = 1, 2, 3$, and ρ is the density of the matter. For static systems the continuity equation is trivially verified and the Euler equation is simplified to

$$\nabla_i p + \rho \nabla_i \phi = 0, \quad (2)$$

where p is the fluid's pressure. Now, a possible enhancement to Newton's theory of gravitation was proposed in² where a relation between the geometry of space and the matter was introduced and given by

$$R = 2k\rho, \quad (3)$$

where R is the Ricci scalar and k is an arbitrary constant. The equations of motion of a test particle with mass m subjected to a gravitational potential ϕ are given by Newton's second law,

$$m a^i = -m g^{ij} \nabla_j \phi, \quad (4)$$

where it was assumed that the inertial and gravitational mass of the test particle are equal. Eqs. (1)-(4) define an enhanced Newtonian gravitation.

3. Spherical Newtonian wormholes

3.1. Construction of the spherical Newtonian wormhole

3.1.1. Matter density and metric

Our purpose is to make use of Eqs. (1)-(3) to construct a static, spherical symmetric wormhole space. Let us then generically write the metric of the space as

$$ds^2 = A(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (5)$$

such that the Ricci scalar R is

$$R = \frac{2[(A(r) - 1)A(r) + rA'(r)]}{r^2 A(r)^2}. \quad (6)$$

Now, to proceed we consider the expression for the mass density to be

$$\rho(r) = \alpha e^{-\frac{r^2}{b^2}} \left(2 - \frac{b^2}{r^2} \right), \quad (7)$$

where α has dimensions of mass density and b of distance. Substituting Eqs. (6) and (7) in Eq. (3) we find the general form of the metric, for the imposed mass density,

$$ds^2 = \frac{r}{r + \beta r e^{-\frac{r^2}{b^2}} + C_1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (8)$$

where C_1 is an integration constant and

$$\beta \equiv b^2 k \alpha. \quad (9)$$

3.1.2. The embedding: the value of the integration constant and the parameter β

Having found the generic form of the metric, we have now to restrict the values of the parameters C_1 , α and k to have a wormhole geometry. For such, we will follow⁶ and use an embedding diagram. So, in the Euclidean embedding space the axially symmetric embedded surface can be, using cylindrical coordinates $(\bar{r}, z, \bar{\varphi})$, uniquely described by a function $z(r)$. Identifying the coordinates $(\bar{r}, \bar{\varphi})$ of the embedding Euclidean space with the coordinates (r, φ) of the wormhole space, we find the following relation

$$\frac{dz}{dr} = \pm \sqrt{\frac{r}{r + \beta r e^{-\frac{r^2}{b^2}} + C_1} - 1}, \quad (10)$$

which can be used to study the properties of the space. From Eq. (10) and imposing the throat condition we find that the integration constant C_1 is given by

$$C_1 = -b \left(1 + \frac{\beta}{e} \right), \quad (11)$$

and the range of values of the parameter $\beta \equiv k \alpha b^2$ is constrained to be $-\infty < \beta < e$. The same restriction to the parameter β is found from the flare-out condition. Substituting Eq. (11) in Eq. (8) we find

$$ds^2 = \frac{1}{1 - \frac{b(r)}{r}} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (12)$$

where

$$b(r) = b + \frac{\beta}{e} \left(b - r e^{1 - \frac{r^2}{b^2}} \right). \quad (13)$$

Another restriction to have a wormhole geometry is that although near the wormhole's throat, $r = b$, the radial coordinate r is ill behaved, the proper radial distance must be finite. This condition implies a new restriction to β

$$-\infty < \beta \leq \beta_{\text{crit}}, \quad (14)$$

where the critical value of β is given by $\beta_{\text{crit}} \equiv \inf_{1 \leq \frac{r}{b} < \infty} \left[\frac{\frac{r}{b} - 1}{e^{-1 - \frac{r}{b}} e^{-\frac{r^2}{b^2}}} \right]$, and \inf represents the infimum of the function in the specified range. This gives

$$\beta_{\text{crit}} = 2.338, \quad (15)$$

up to the third decimal place. Comparing Eq. (15) with the restriction found from the throat condition, $\beta < e$, we conclude that the product $\beta = k\alpha b^2$ is then restricted by Eqs. (14) and (15).

Due to the range of values that the parameter β might take, various distinct cases could be considered. Here we shall be interested in the class of wormholes whose parameters α , k and b obey $0 < \beta \leq \beta_{\text{crit}}$ together with $\alpha > 0$, and so $k > 0$.

3.1.3. Removal of the coordinate singularity

Now that we have found the metric of the Newtonian wormhole we have to remove the coordinate singularity at $r = b$. Considering a new coordinate l , defined as $l^2 = r^2 - b^2$, such that $-\infty < l < \infty$, one finds that the metric of the space can be rewritten as

$$ds^2 = \frac{e l^2}{(b^2 + l^2) \left(e + \beta e^{-\frac{l^2}{b^2}} \right) - b\sqrt{b^2 + l^2} (e + \beta)} dl^2 + (l^2 + b^2) (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (16)$$

It will also be useful to write the mass density in the new coordinates. It is given by the expression

$$\rho(l) = \alpha \frac{e^{-1 - \frac{l^2}{b^2}} (b^2 + 2l^2)}{b^2 + l^2}. \quad (17)$$

3.2. Gravitational field, gravitational potential and pressure support of the spherical Newtonian wormhole

Having defined the geometry of the space we can now study the gravitational potential of the Newtonian wormhole by solving Eq. (1). It is, however, considering the symmetries of the system, straightforward to solve Eq. (1) by using Gauss's law for gravity and integrating this equation over a volume whose boundary are the surfaces of constant gravitational potential. Defining the gravitational force field as

$$\mathcal{G}^i \equiv g^{ij} \phi_{,j}, \quad (18)$$

with a comma denoting a partial derivative, we find

$$\mathcal{G}^i(l) = \frac{G m(l)}{2(b^2 + l^2)} \sqrt{\frac{(b^2 + l^2) \left(e + \beta e^{-\frac{l^2}{b^2}} \right) - b\sqrt{b^2 + l^2} (e + \beta)}{e l^2}} \delta_l^i, \quad (19)$$

where the mass within radius l is given by

$$m(l) = 4\pi\alpha b^3 \int_1^{1+l^2/b^2} \frac{e^{-x}(2x-1)}{\sqrt{x(1+\beta e^{-x}) - \left(1 + \frac{\beta}{e}\right)\sqrt{x}}} dx. \quad (20)$$

Defining the magnitude of the gravitational field as

$$\mathcal{G} = \sqrt{g_{ij} \mathcal{G}^i \mathcal{G}^j}, \quad (21)$$

we give in Fig. 1 its behavior as a function of the radial coordinate l for various values of the parameter β .

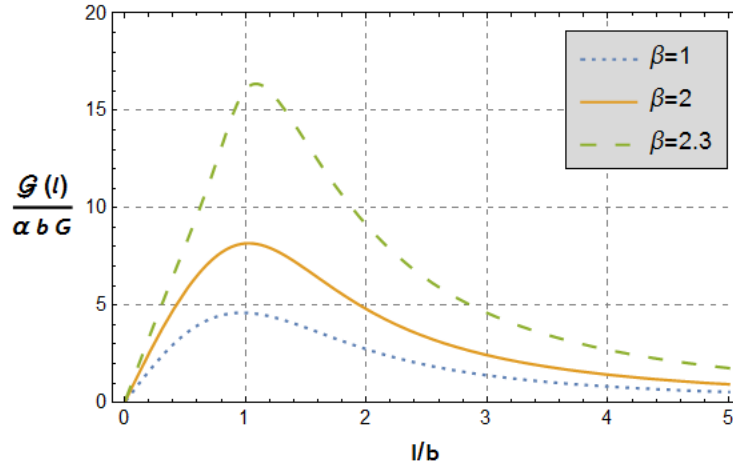


Fig. 1. Magnitude of the gravitational force field for various values of the parameter $\beta \equiv b^2 k \alpha$ as a function of the radial coordinate l (here assume $l \geq 0$). When $\beta = \beta_{\text{crit}} = 2.338$ the magnitude of the gravitational force field tends to infinity at some value of l/b .

Using the expression for the gravitational field, Eq. (19) together with Eq. (18), the gravitational potential ϕ can be computed at a given point using numerical methods, provided the wormhole parameters are given. Moreover, these equations can also be used to find the pressure of the fluid that covers the wormhole space. Numerically solving Eqs. (2) and (19) we find that the pressure is positive throughout the space showing that the wormhole is hold against gravitational collapse by pressure⁵. From Eq. (18) one sees that the gradient of the potential ϕ is the gravitational force field \mathcal{G}^i , and from Eq. (2) the gradient of the pressure p goes with minus the gradient of the gravitational potential ϕ . Thus one can infer from Fig. 1, and the help of Eqs. (2), (18), and (21), that at the wormhole's center $l = 0$, i.e., at the throat, the pressure is maximal, as one would expect.

In the limiting case that the wormhole has $\beta = \beta_{\text{crit}}$ a Newtonian event horizon develops at some radial coordinate $l_h > 0$, in the sense that any particle inside the sphere defined by $l_h > 0$ can only affect an outside observer if it has infinite

acceleration, and thus infinite velocity. Thus, since no particles can come out of this inner region one is in the presence of a Newtonian black hole. However, for $\beta = \beta_{\text{crit}}$ the pressure support at the throat goes to infinity, and so, in this limit, the wormhole eventually collapses.

3.3. *Test particle dynamics and motion in the spherical wormhole gravitational field*

3.3.1. *The equations of motion*

Now we use Eq. (4) to study the motion of a test particle in the gravitational field of the wormhole space. Consider that a particle's path is described by a curve γ , whose components x^i are given by $x^i = (l(t), \theta(t), \varphi(t))$ such that the particle's velocity is $v^i \equiv \dot{x}^i$, where a dot means a derivative with respect to time t . The right hand side of Eq. (4) is the gravitational field given in Eq. (19) and the left hand side the acceleration of the particle, defined as $a^i = v^j \nabla_j v^i$. It is possible to show that one can treat the problem of the motion of a test particle by considering pure equatorial orbits, i.e., $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$ and $\ddot{\theta} = 0$. Gathering these results the equations of motion are

$$\begin{aligned} \ddot{l} + \left[\frac{b e^{\frac{l^2}{b^2}} (2b^2 + l^2) (e + \beta) - 2\sqrt{b^2 + l^2} \left[\beta l^2 (1 + l^2/b^2) + b^2 \left(e^{1 + \frac{l^2}{b^2}} + \beta \right) \right]}{2l (b^2 + l^2) \left[b e^{\frac{l^2}{b^2}} (e + \beta) - \sqrt{b^2 + l^2} \left(e^{1 + \frac{l^2}{b^2}} + \beta \right) \right]} \right] \dot{l}^2 + \\ + \left[\frac{b\sqrt{b^2 + l^2} (e + \beta) - (b^2 + l^2) \left(e + \beta e^{-\frac{l^2}{b^2}} \right)}{e l} \right] \dot{\varphi}^2 = \\ = -\frac{G m(l)}{2\sqrt{e} (b^2 + l^2) l} \sqrt{(b^2 + l^2) \left(e + \beta e^{-\frac{l^2}{b^2}} \right) - b\sqrt{b^2 + l^2} (e + \beta)}, \end{aligned} \quad (22)$$

$$\theta = \frac{\pi}{2}, \quad \dot{\theta} = 0, \quad (23)$$

$$\ddot{\varphi} + 2 \frac{l}{b^2 + l^2} \dot{l} \dot{\varphi} = 0. \quad (24)$$

3.3.2. *Solutions of the equations of motion*

Having found the equations of motion of a test particle, we have now to solve them. Let us consider the simpler problem of a test particle describing pure circular motion in the Newtonian wormhole space. In this case the coordinate l is a constant, l_0 , say, so $\dot{l} = 0$ and $\ddot{l} = 0$. From this, Eq. (22) can be solved for $\dot{\varphi}$ and obtain

$$\dot{\varphi} = \sqrt{\frac{\sqrt{e} G m(l_0)}{2 (b^2 + l_0^2) \sqrt{(b^2 + l_0^2) \left(e + \beta e^{-\frac{l_0^2}{b^2}} \right) - b\sqrt{b^2 + l_0^2} (e + \beta)}}}. \quad (25)$$

Eq. (25) relates the radial position l_0 and the angular velocity $\dot{\varphi}$ for a particle to describe a circular orbit in the wormhole space. Now, except the case of pure circular motion, to solve the equations of motion (22)-(24) for a more generic motion, numerical methods must be used, provided the initial position and velocity of the test particle and the wormhole parameters b and α are given.

3.4. Tidal effects from the gravitational field and the geometry of the spherical Newtonian wormhole

3.4.1. Tidal deviation equation for the spherical Newtonian wormhole

We now analyze the relative separation of two test particles in the gravitational field of the spherical Newtonian wormhole space. The two test particles are initially considered to be infinitesimally close, subjected only to the gravitational field of the Newtonian wormhole, such that the acceleration of a test particle at a certain point is given by Eq. (4) and Eqs. (19)-(20). This relative separation, i.e., the tidal deviation equation, is in part given by the variation of the gravitational force field acted on each particle and another part given by the geodesic deviation due to the spatial curvature through the equation,

$$\frac{D^2 n^i}{dt^2} = n^k (-\nabla_k \mathcal{G}^i + R_{jlk}^i v^j v^l), \quad (26)$$

where n^i is the separation vector between two infinitesimally close particles, \mathcal{G}^i is the gravitational force field generated by the wormhole mass, given by Eq. (19), v^i is the velocity of the fiducial test particle and R_{jlk}^i represents the Riemann tensor.

Let us simplify the calculations, by remarking that, as was mentioned in Sec. (3.3), the velocity of the fiducial test particle can be assumed to be along the equator, $\theta = \pi/2$ (see Eq. (23)). Hence, in Eq. (26), we might take all the terms in v^θ to zero. Notice, however, that this does not mean that, in general, the acceleration of the component n^θ of the separation vector is zero since the motion of the second particle may not be along the equator. Now, considering the metric of the spherical Newtonian wormhole space, Eq. (16), and assuming $\theta = \pi/2$ and $\dot{\theta} = 0$, we find from Eq. (26) that the deviation equations for two infinitesimally close test particles in the spherically symmetric Newtonian wormhole space are

$$\frac{D^2 n^l}{dt^2} = -n^l (\partial_l \mathcal{G}^l + \Gamma_{ll}^l \mathcal{G}^l) + R_{\varphi l \varphi}^l v^\varphi (n^\varphi v^l - n^l v^\varphi), \quad (27)$$

$$\frac{D^2 n^\theta}{dt^2} = n^\theta \left[-\Gamma_{\theta l}^\theta \mathcal{G}^l + R_{ll\theta}^\theta (v^l)^2 + R_{\varphi\varphi\theta}^\theta (v^\varphi)^2 \right], \quad (28)$$

$$\frac{D^2 n^\varphi}{dt^2} = -\Gamma_{\varphi l}^\varphi \mathcal{G}^l n^\varphi + R_{ll\varphi}^\varphi v^l [n^\varphi v^l - n^l v^\varphi]. \quad (29)$$

where the Christoffel symbols that appear in Eqs. (27)-(29) have the form

$$\Gamma_{ll}^l = \frac{b^3(\beta + e)e^{\frac{l^2}{b^2}}(2b^2 + l^2) - 2\sqrt{b^2 + l^2} \left[l^2\beta(b^2 + l^2) + b^4 \left(e^{\frac{l^2}{b^2} + 1} + \beta \right) \right]}{2b^2l(b^2 + l^2) \left[b(\beta + e)e^{\frac{l^2}{b^2}} - \sqrt{b^2 + l^2} \left(e^{\frac{l^2}{b^2} + 1} + \beta \right) \right]}, \quad (30)$$

$$\Gamma_{\theta l}^{\theta} = \Gamma_{\varphi l}^{\varphi} = \frac{l}{b^2 + l^2}. \quad (31)$$

The other non-zero Christoffel symbols necessary to calculate the Riemann tensor components are $\Gamma_{\varphi\varphi}^l = \Gamma_{\theta\theta}^l \sin^2 \theta$, $\Gamma_{\varphi\theta}^{\varphi} = \cot \theta$, and $\Gamma_{\varphi\varphi}^{\theta} = -\Gamma_{\varphi\theta}^{\varphi} \sin^2 \theta$. At the plane $\theta = \pi/2$ one has

$$\Gamma_{\theta\theta}^l = \Gamma_{\varphi\varphi}^l = -\frac{(b^2 + l^2) \left(\beta e^{-\frac{l^2}{b^2}} + e \right) - b(\beta + e)\sqrt{b^2 + l^2}}{el}, \quad (32)$$

and $\Gamma_{\varphi\theta}^{\varphi} = \Gamma_{\varphi\varphi}^{\theta} = 0$, the remaining Christoffel symbols being zero or given by symmetry of the ones just calculated. The Riemann tensor components appearing in Eqs. (27)-(29), assuming $\theta = \pi/2$, are

$$R_{\varphi l \varphi}^l = \frac{2\beta e^{-\frac{l^2}{b^2}} \sqrt{(b^2 + l^2)^3} - b^3(\beta + e)}{2eb^2 \sqrt{b^2 + l^2}}, \quad (33)$$

$$R_{ll\theta}^{\theta} = R_{ll\varphi}^{\varphi} = \frac{l^2 \left(2\beta \sqrt{(b^2 + l^2)^3} - b^3(\beta + e) e^{\frac{l^2}{b^2}} \right)}{2b^2(b^2 + l^2)^2 \left(b(\beta + e) e^{\frac{l^2}{b^2}} - \sqrt{b^2 + l^2} \left(e^{\frac{l^2}{b^2} + 1} + \beta \right) \right)}, \quad (34)$$

$$R_{\varphi\varphi\theta}^{\theta} = \frac{\beta e^{-\frac{l^2}{b^2}} \sqrt{b^2 + l^2} - b(\beta + e)}{e \sqrt{b^2 + l^2}}. \quad (35)$$

These equation complete the geodesic deviation equations (27)-(29).

3.4.2. Tidal effects for pure radial motion

Now, looking at Eqs. (27)-(29), in general, it is not possible to find analytical solutions since the fiducial test particle's velocity must also be taken into account. So, to solve this system of differential equations, we must first solve the equations of motion, Eqs. (22)-(24), and in general there is no analytical solution. We can however work out some features of the tidal effects in the case of pure radial motion of two test particles such that throughout the motion the line that connects the two particles is purely radial, i.e., the case where n^{θ} , n^{φ} , v^{θ} , and v^{φ} are zero throughout the particles' motion. In this case Eq. (27) simplifies to

$$\frac{D^2 n^l}{dt^2} = -n^l (\partial_l \mathcal{G}^l + \Gamma_{ll}^l \mathcal{G}^l), \quad (36)$$

and Eqs. (28) and (29) yield $\frac{D^2 n^{\theta}}{dt^2} = 0$ and $\frac{D^2 n^{\varphi}}{dt^2} = 0$, respectively. Although it is still not possible to solve analytically Eq. (36) we can instead make a qualitative analysis to infer the general behavior of two initially close test particles describing pure radial motion.

Defining the right hand side of Eq. (36) as

$$\Delta(l) \equiv -\partial_l \mathcal{G}^l - \Gamma_{ll}^l \mathcal{G}^l \quad (37)$$

we present in Fig. 2 its behavior. We see that the sign of the function Δ changes at some value of the coordinate l , l_c , say. Now, when $l > l_c$ the function $\Delta(l)$ is positive,

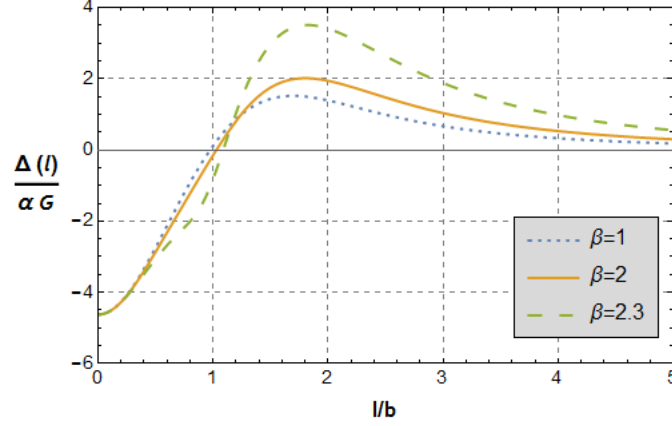


Fig. 2. Plot of the quantity $\Delta \equiv -\partial_l \mathcal{G}^l - \Gamma_{\mu}^l \mathcal{G}^{\mu}$ as a function of the radial coordinate l (here assume $l \geq 0$).

$\Delta(l) > 0$, which implies that the separation vector has positive acceleration. So, suppose two test particles describing pure radial motion, such that the line that connects them is purely radial. One particle has radial coordinate l_1 , the other l_2 , and let us assume, $l_1 < l_2$ and $l_{1,2} > l_c$ with $l_c > 0$ for simplicity. In this case, since $\Delta(l) > 0$, the particle with coordinate l_1 is accelerating more than the particle with coordinate l_2 , and they fly apart. On the other hand, for $l < l_c$ the function $\Delta(l)$ is negative, $\Delta(l) < 0$, so the separation vector has negative acceleration. Assuming now $l_{1,2} < l_c$, and still $l_1 < l_2$, this means that the particle with coordinate l_1 is accelerating less, but still towards the wormhole's throat, than the particle with coordinate l_2 . Numerically solving Eq. (36) for two initially close test particles in the regions $l > l_c$ and $l < l_c$, indeed, we verify this conclusions. Notice that as we consider particles further away from the wormhole's throat we recover the tidal behavior expected from Newtonian gravitation since the space is asymptotically flat.

4. Conclusions

A static, spherically symmetric wormhole in an enhanced Newtonian theory of gravitation was constructed. The Newtonian wormhole's mass density is positive, the gravitational field of the wormhole is well-behaved and the matter that sustains it has positive pressure. Test particles' motion in the wormhole gravitational field and tidal effects were studied. We have also argued about the possibility of having true Newtonian black holes, i.e., Newtonian objects that have regions from which any particle must have infinite acceleration and thus infinite velocity to escape to the outside of it, in this enhanced Newtonian theory of gravitation.

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